ASYMPTOTIC SOLUTION OF THE NAVIER–STOKES EQUATIONS FOR RADIAL FLUID FLOW IN THE GAP BETWEEN TWO ROTATING DISKS

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The known analytical particular solutions of the Navier–Stokes equations for the motion of a fluid between two disks rotating with the same [1, 2] or different [3] angular velocities are expressed as power series of two dimensionless parameters, one of which is proportional to the angular velocity of rotation, and the other to the radial velocity at a given radius averaged over the gap. In the present paper the solution is expressed as a power series in the second of these dimensionless parameters only. The coefficients of the various powers of the parameter, which are functions of the transverse coordinates and also of a parameter characterizing the intensity of the rotational motion, are found by numerical integration of a set of ordinary differential equations. To zero order, the resulting solution reduces to the well-known self-modeling numerical solutions of the Navier–Stokes equations for the case of zero net radial flow between the two rotating disks [4, 5]. The solution is useful for analysis of flow under a wider range of the parameters of the problem compared to the solutions of [1-3]. The results of the solution and comparison to experimental data [6-7] are presented in detail for the case when the two rotating disks have the same velocity.

1. We study the axisymmetric laminar flow of a viscous, incompressible fluid inside the narrow gap between two parallel disks, which in general rotate with different velocities about the same axis. The origin of a cylindrical coordinate system r, θ, z is placed at the intersection of the axis of rotation with the inner surface of one of the disks, whose angular velocity of rotation is set equal to ω. The inner surface of the second disk lies at $z = h$ and it rotates with a constant angular velocity $\alpha\omega$ $(-1 \le \alpha \le 1)$. Inside the gap, on the rotation axis, is a source (or sink) of fluid of given output $2\pi Q$. The motion of the fluid in the gap is described by the Navier–Stokes equations

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right),$$
$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right), \quad (1.1)$$
$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right), \quad \frac{\partial (ru)}{\partial r} + \frac{\partial (rw)}{\partial z} = 0$$

and the following boundary conditions with respect to the z direction:

$$u = 0,\ v = \omega r,\ w = 0 \ \text{for} \ z = 0,$$
$$u = 0,\ v = \alpha\omega r,\ w = 0 \ \text{for} \ z = h, \quad (1.2)$$

where u, v, w are the r, θ, z components of the velocity, and p, ρ, ν are the pressure, density, and coefficient of kinematic viscosity of the fluid.

The semianalytical method of constructing a particular solution of (1.1) used in our paper does not require boundary conditions with respect to the r coordinate. The applicability of the solution to flow analysis for arbitrary distributions $u(z)$, $v(z)$ on the cylindrical surface bounding the gap is discussed below.

The equation of continuity and the above boundary conditions give the following relation

$$\int_0^h ru\,dz = Q = \text{const}. \quad (1.3)$$

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2. We now introduce the flow function

$$u = \frac{1}{r}\frac{\partial\psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial\psi}{\partial r} \tag{2.1}$$

and the dimensionless variables

$$\xi = z/h, \quad \psi = -\frac{1}{2}\omega r^2 hH,$$
$$u = -\frac{1}{2}\omega rU = -\frac{1}{2}\omega r\frac{\partial H}{\partial\xi}, \quad v = \omega rG. \tag{2.2}$$

We substitute (2.1) and (2.2) into (1.1) and the conditions (1.2), (1.3). Next we transform to the new argument ε related to the radial coordinate by $\varepsilon = 2Q/\omega r^2h$. Using the relations

$$r\frac{\partial}{\partial r} = -2\varepsilon\frac{\partial}{\partial\varepsilon}, \quad r^2\frac{\partial}{\partial r^2} = 2\varepsilon\frac{\partial}{\partial\varepsilon} + 2\varepsilon\frac{\partial}{\partial\varepsilon}\left(2\varepsilon\frac{\partial}{\partial\varepsilon}\right),$$

we obtain the following equations

$$U'' - R\left(HU' + 2G^2 - \frac{1}{2}U^2\right) = 2R\Pi + R\varepsilon\left(U\frac{\partial U}{\partial\varepsilon} + U'\frac{\partial H}{\partial\varepsilon}\right) + O\left(\frac{h^2}{r^2}\varepsilon\right),$$
$$G'' + R(GU - HG') = R\varepsilon\left(U\frac{\partial G}{\partial\varepsilon} - G'\frac{\partial H}{\partial\varepsilon}\right) + O\left(\frac{h^2}{r^2}\varepsilon\right),$$
$$U = H', \quad p = p^* + O\left(\frac{h^2}{r^2}\varepsilon\right). \tag{2.3}$$

The boundary conditions for (2.3) have the form

$$U = 0, \ G = 1, \ H = 0 \ \text{for} \ \xi = 0, \ U = 0, \ G = \alpha, \ H = -\varepsilon \ \text{for} \ \xi = 1, \tag{2.4}$$

where the prime denotes differentiation with respect to ξ and $R = \omega h^2/\nu$ is a basic parameter of the problem. The function $\Pi = -\frac{1}{\rho\omega^2 r}\frac{dp^*}{dr}$, where $p^* = \int_0^1 p\,d\xi$, is determined with the help of the boundary condition for the dimensionless flow function $H(\xi = 1) = -\varepsilon$. The dimensionless quantities U, G, H are all of order unity. Putting $(h/r)^2\varepsilon \ll 1$, we omit terms of this order in (2.3) in the calculations below.

The point $\varepsilon = 0$ is a singular point of Eq. (2.3) at which they reduce to equations used in finding self-modeling solutions for flow either close to the rotating surface or boundary-layer motion when the fluid far from the surface rotates as a solid.

We emphasize that ε can be made arbitrarily small, not only by reducing the output of the source Q, but also for a fixed finite value of Q if ω or r become large enough.

The solutions $H_0(\xi)$, $U_0(\xi)$, $G_0(\xi)$ of (2.3) at $\varepsilon = 0$ are completely determined by the boundary conditions (2.4). It would appear that this solution could be used as a starting point for the integration of (2.3), (2.4). However, when $\varepsilon \geq 0$, one can always find for any α a region in which U becomes negative at least over part of the range of ξ, and this results in an incorrectly posed problem [8]. Numerical integration of (2.3) with H_0, U_0, G_0 as a starting point is only possible when $\varepsilon < 0$, and for bounded ranges of the parameters α and R.

The semianalytical method is used to solve (2.3), (2.4). We expand the functions to be determined in power series in ε and numerically integrate the resulting set of ordinary differential equations. The solution of (2.3), (2.4) is thus constructed over a wide range of the argument ε and the parameters α and R. The radius of convergence of the series is not determined, but the scale behavior of the coefficients of increasing powers of ε suggest that the series converges rapidly, at least when R is not too large. The non-self-modeling nature of the problem is allowed for in the net action of the source (sink). Other factors that would cause the problem to be nonself-modeling, such as specified distributions of U and G on the inflow (or outflow) from the gap, vanish with distance (the scale factor is of the order of the gap separation) whereas the source (sink) is felt more or less over the entire gap. Hence when $h/r \ll 1$, the flow characteristics for arbitrary boundary conditions asymptotically approach the (particular but in practical dominant) solution discussed here upon displacement from the edge of the gap to the interior.

3. The solution is expanded in the form

$$F = F_0 + F_1\varepsilon + F_2\varepsilon^2 + F_3\varepsilon^3 + \ldots, \tag{3.1}$$

where F represents any of the functions H, U, G, Π.

Expansion (3.1) is substituted into (2.3) with terms up to order ε^2 inclusive being kept. We then obtain four sets of differential equations with corresponding boundary conditions. The equations and boundary conditions defining the lowest order approximation are obtained easily from (2.3) and (2.4) with $\varepsilon = 0$. For the higher order approximations, we introduce the general notation

$$U_i'' - RH_0U_i' + A_i^1U_i = B_i^1,\ G_i'' - RH_0G_i' + A_i^2G_i = B_i^2,\ U_i = H_i',$$

where the coefficients of the functions are given in Table 1, as well as expressions for the right-hand sides of these equations. All boundary conditions now have the form $U_i = G_i = H_i = 0$ at $\xi = 0$ and $\xi = 1$ except for the condition $H_1 = -1$ at $\xi = 1$;

The above ordinary differential equations were integrated successively by numerical methods. We used the center approximation on a uniform grid for the first derivatives, and the usual three-point approximation for the second derivatives. The method of adjustment was then applied to the solution. For each time step the set of algebraic equations were solved by the trial run method for complex systems [9]. The adjustment process was considered completed when the condition $\sum_{m=0}^{M} |F_{im}^{k+1} - F_{im}^{k}| / \sum_{m=0}^{M} |F_{im}^{k+1}| \leqslant \delta$, was reached where F_i is any of the functions H_i, U_i, G_i (i = 0, 1, 2, 3); m is the number of vertices of the grid; M is the total number of vertices; k is the number of the time step.

Systematic calculations were done for M = 50 and $\delta = 10^{-4}$. The results of the solution in the zero order approximation for $\alpha = 0$ (rotating and fixed disks) agrees with [4, 5] up to R = 100. The principal calculations were done for the case $\alpha = 1$ (both disks rotating with the same velocity) where $U_0 = 0$, $G_0 = 1$. As an example, in Fig. 1 results of the calculation of the functions F_i in the expansion (3.1) for the axial and circular components of the velocity are given for R = 16.0. Curves 1-3 correspond to U_i where i = 1, 2, 3, and curves 4-6 to G_i. It can be seen that even for large values of R, the functions F_i not only do not increase with i but show a marked decrease. The most interesting integral characteristics of the flow from a practical point of view are given in Table 2.

In [7] the components of the velocity of the fluid were measured (for the case $\alpha = 1$) for flow into an aperture close to the center of the disks. The experimental [7] and calculated radial velocity profiles are shown in Fig. 2 for R = 4.0, $\varepsilon = 0.0304$ (curve 1); R = 9.0, $\varepsilon = 0.0203$ (curve 2); R = 16.0, $\varepsilon = 0.0152$ (curve 3). The excellent agreement between the experimental and calculated data can be seen. Comparison of the results for the circular component of the velocity leads to the same conclusion.

Measurements of the pressure distribution along the gap for fluid supply from the periphery of the disks

TABLE 1

i	1	2	3
A_i^1	0	$-RU_0$	$-2RU_0$
A_i^2	0	$-RU_0$	$-2RU_0$
B_i^1	$2R\Pi_1+4RG_0G_1$	$2R\Pi_2+R\left[\frac{1}{2}U_1^2-H_2U_0'+ +2(G_1^2+2G_0G_2)\right]$	$2R\Pi_3+R(2U_1U_2-U_1'H_2+4G_1G_2+ +4G_0G_3-2U_0'H_3)$
B_i^2	$-RU_1G_0$	$-R(G_0'H_2+G_0U_2)$	$-R(U_1G_2+U_3G_0+2G_0'H_3+G_1'H_2)$

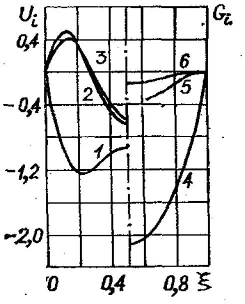

Fig. 1

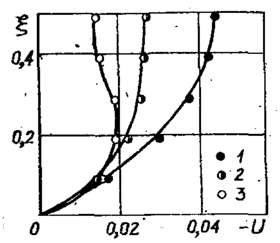

Fig. 2

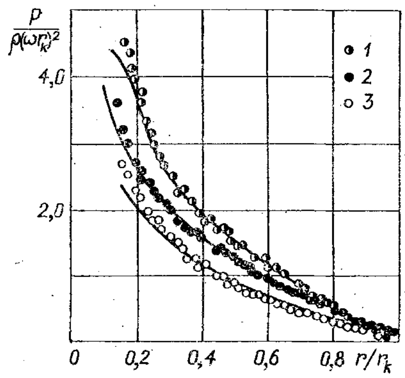

Fig. 3

TABLE 2

R	1,0	2,0	4,0	9,0	16,0	25,0	49,0
Π_0	—1,0	—1,0	—1,0	—1,0	—1,0	—1,0	—1,0
Π_1	6,2407	3,4833	2,4618	2,7578	3,8034	4,9598	7,0321
Π_2	—0,4002	—0,4459	—0,6202	—1,4343	—2,9205	—4,6150	—8,5520
Π_3	—0,0029	—0,0080	—0,0254	—0,1916	—0,9373	—2,2767	—4,5398
$G_0'(0)$	0	0	0	0	0	0	0
$G_1'(0)$	—0,4987	—0,9981	—1,9989	—4,5019	—8,0109	—12,5322	—24,6459
$G_2'(0)$	0,0004	0,0017	0,0027	0,0039	0,0109	0,0322	0,1764
$G_3'(0)$	0,0000	—0,0002	—0,0008	0,0004	0,0041	0,0240	0,1573

[6] is shown in Figs. 3 ($R = 4.29$; $\varepsilon_k = -0.0468$ (curve 1); $R = 2.17$, $\varepsilon_k = -0.319$ (curve 2); $R = 7.11$, $\varepsilon_k = -0.280$ (curve 3); where the index k refers to quantities evaluated at the outer rim of the disks where $r = r_k$). The calculated curves (solid curves) deviate from the experimental data only close to the center of the aperture. We note that even for extremely large local values of $\varepsilon = (r/r_k)^{-2}\varepsilon_k$ where for $r/r_k = 0.2$ we have $R = 4.29$, $\varepsilon = 11.7$; $R = 2.17$, $\varepsilon = 7.98$; $R = 7.11$, $\varepsilon = 7.00$, the agreement between the calculated and experimental results is good. However, further increase of ε (decrease of r) leads to a sharp divergence between the two sets of results. Defining the value of ε (for a given R) corresponding to the start of the sharp deviation of the calculated value of dp^*/dr from the experimental value, one can estimate the region of applicability of our solution for the pressure distribution: $R|\varepsilon| < 50$.

Comparison of our solution with the results in [10] of numerical integration of the equations of motion for flow from the periphery also shows the good agreement of the results at far enough distance from the entry into the gap. The flow in a narrow gap at the entry region is analyzed approximately in [11]. Comparison of the data of [10, 11] with the results of our solution shows that the effect of the initial distribution for $R|\varepsilon| < 10$ vanishes for flow from the center when $r/r_i > 1.2$ (r_i is the radius of the inner rim of the disk) and for flow from the periphery when $r/r_k < 0.9$. These inequalities must obviously be taken only as estimates.

Finally, calculations for $\alpha = 0$ and -1 did not lead to any additional difficulties.

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STATIONARY LONG WAVES IN A LIQUID FILM ON AN INCLINED PLANE

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The nonlinear equations describing wave flow of a thin liquid film are normally obtained with the aid of assumptions as to the character of the distribution of the transverse velocity component over the film thickness. Such an approach was used with a system of two equations for liquid flow rate and film thickness deviation from the value corresponding to nonwave laminar flow in [1-3]. In [4-6] a unique evolution equation for film thickness was also obtained with a method similar to the conventional Karman–Polhausen technique. In this case the question of the range of applicability of the equation obtained and the accuracy of its description of the wave process arises. To answer this question one must obviously use direct methods to derive the evolution equation, with simultaneous definition of the velocity profile within the film [7-9]. This will be done below for small Reynolds numbers for a flow on an inclined plane (considered previously in [6, 10, 11]). One of the equations obtained is suitable for study of slightly nonlinear stationary traveling waves. In contrast to previous studies of stationary regimes, all parameters of such waves are defined uniquely.

1. Flow in the Film. We introduce the dimensionless variables and parameters

$$t=\frac{u_0}{\lambda}t',\ x=\frac{x'}{\lambda},\ y=\frac{y'}{h_0},\ \begin{Bmatrix}v_x\\ v_y\end{Bmatrix}=\frac{1}{u_0}\begin{Bmatrix}v_x'\\ v_y'\end{Bmatrix},\tag{1.1}$$

$$\varphi=\frac{h-h_0}{h_0},\ p=\frac{\mathrm{Re}}{\rho u_0^2}p',\ \mathrm{Re}=\frac{u_0h_0}{\nu},\ \varepsilon=\frac{h_0}{\lambda},$$

$$T=\frac{3\varepsilon^3\,\mathrm{We}}{\cos\alpha},\ \mathrm{We}=\frac{\sigma}{\rho g h_0^2},\ u_0=\left(\frac{\cos\alpha}{3}\frac{g}{\nu}\right)^{1/3}Q^{2/3},\ h_0=\left(\frac{3}{\cos\alpha}\frac{\nu Q}{g}\right)^{1/3}.$$

Here the primes denote the corresponding dimensional variables, α is the angle of inclination of the plate to the vertical, λ is the characteristic longitudinal scale, u_0 and h_0 are the mean film velocity and thickness in the nonwave regime. The equations describing the motion written in the variables of Eq. (1.1) have the form

$$\varepsilon\,\mathrm{Re}\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}-\int_0^y\frac{\partial v_x}{\partial x}dy\,\frac{\partial v_x}{\partial y}\right)=\frac{\partial^2 v_x}{\partial y^2}+\varepsilon^2\frac{\partial^2 v_x}{\partial x^2}-\tag{1.2}$$

$$-\varepsilon\frac{\partial p}{\partial x}+3,\ \varepsilon\,\mathrm{Re}\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}-\int_0^y\frac{\partial v_x}{\partial x}dy\,\frac{\partial v_y}{\partial y}\right)=\frac{\partial^2 v_y}{\partial y^2}+\varepsilon^2\frac{\partial^2 v_y}{\partial x^2}-\frac{\partial p}{\partial y}-3\,\mathrm{tg}\,\alpha,\ \frac{\partial v_y}{\partial y}=-\varepsilon\frac{\partial v_x}{\partial x}.$$

The boundary conditions for Eq. (1.2) at y = 0 have the form

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